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1999 J. Phys. A: Math. Gen. 32 7933

(http://iopscience.iop.org/0305-4470/32/45/309)

J. Phys. A: Math. Gen. **32** (1999) 7933–7952. Printed in the UK PII: S0305-4470(99)05160-4

Painlevé test and the first Painlevé hierarchy

U Muğan and F Jrad
Department of Mathematics, Bilkent University, 06533 Bilkent, Ankara, Turkey
E-mail: mugan@fen.bilkent.edu.tr

Received 14 June 1999

Abstract. Starting from the first Painlevé equation, Painlevé type equations of higher order are obtained by using the singular point analysis.

1. Introduction

Painlevé and his school [1–3] around the turn of the century investigated the second-order equations of the form

$$y'' = F(z, y, y') \tag{1.1}$$

where F is rational in y', algebraic in y and locally analytic in z, and have no movable critical points, i.e., the location of the singularities of any of the solutions other than poles depend only on the equation. This property is known as the Painlevé property and ordinary differential equations (ODEs) which possess it are said to be of Painlevé type. Within the Möbius transformation, they found 50 such equations. Distinguished amongst these 50 equations are the six Painlevé equations $P_{\rm I}, P_{\rm II}, \ldots, P_{\rm VI}$; any of the other 44 equations can either be integrated in terms of the known functions or can be reduced to one of the six equations. Although the Painlevé equations were discovered from strictly mathematical considerations, they have appeared in many physical problems, and possess rich internal structure.

The Riccati equation is the only example for the first-order first-degree equation which has the Painlevé property. Before the work of Painlevé and his school, Fuchs [3, 4] considered the equation of the form

$$F(z, y, y') = 0 \tag{1.2}$$

where F is a polynomial in y and y' and locally analytic in z, such that the movable branch points are absent, that is, the generalization of the Riccati equation. The irreducible form of the first-order algebraic differential equation of the second degree is

$$a_0(z)(y')^2 + \sum_{i=0}^{2} b_i(z) y^i y' + \sum_{j=0}^{4} c_j(z) y^j = 0 \tag{1.3}$$

where b_i, c_j are analytic functions of z and $a_0(z) \neq 0$. Briot and Bouquet [3] considered the subcase of (1.2). That is, first-order binomial equations of degree $m \in Z_+$:

$$(y')^m + F(z, y) = 0 \tag{1.4}$$

0305-4470/99/457933+20$30.00

where $F(z, y)$ is a polynomial of degree at most $2m$ in y. It was found that there are six types of equation of the form (1.4). But all these equations are either reducible to a linear equation or solvable by means of elliptic functions [3].

Second-order second-degree Painlevé-type equations of the following form:

$$(y'')^2 = E(z, y, y')y'' + F(z, y, y') \tag{1.5}$$

where E and F are assumed to be rational in y, y' and locally analytic in z were the subject of the papers [5–7]. In [5, 6], the special form, $E = 0$, and hence F is polynomial in y and y' of (1.5) was considered. Also in this case no new Painlevé-type equation was discovered, since all of them can be solved either in terms of the known functions or one of the six Painlevé equations. In [7], it was shown that all the second-degree equations obtained in [5, 6], the $E = 0$ case, and some of the second-degree equations such that $E \neq 0$ can be obtained from $P_{\mathrm{I}}, \ldots, P_{\mathrm{VI}}$ by using the following transformations which preserve the Painlevé property:

$$u(z, \hat{\alpha}) = \frac{y' + \sum_{i=0}^{2} a_i y^i}{\sum_{i=0}^{2} b_i y^i} \tag{1.6}$$

and

$$u(z, \hat{\alpha}) = \frac{(y')^2 + \sum_{i=0}^{2} a_i(z) y^i y' + \sum_{j=0}^{4} b_j(z) y^j}{\sum_{i=0}^{2} c_i(z) y^i y' + \sum_{j=0}^{4} d_j(z) y^j} = 0 \tag{1.7}$$

where a_i, b_j, c_i, d_j are analytic functions of z. That is, if y solves one of the Painlevé equation with parameter set α then u solves a second-order second-degree Painlevé-type equation of the form (1.5).

The special form, of polynomial type, of the third-order Painlevé-type equations

$$y''' = F(z, y, y', y'') \tag{1.8}$$

was considered in [8,9]. The most well known third-order equation is Chazy's 'natural-barrier' equation

$$y''' = 2yy'' - 3y'^2 + \frac{4}{36 - n^2}(6y' - y^2)^2. \tag{1.9}$$

The case $n = \infty$ appears in several physical problems. Equation (1.9) is integrable for all real and complex n and $n = \infty$. Its solutions are rational for $2 \leqslant n \leqslant 5$, and have a circular natural barrier for $n \geqslant 7$ and $n = \infty$. Bureau [9] considered the third-order equation of Painlevé type of the following form:

$$y''' = P_1(y)y'' + P_2(y)y'^2 + P_3(y)y' + P_4(y) \tag{1.10}$$

where $P_n(y)$ is a polynomial in y of degree n with analytic coefficients in z. Also in [9] were some of the fourth-order polynomial-type equations

$$y^{(4)} = ayy''' + by'y'' + cy^2y'' + dyy'^2 + ey^3y' + fy^5 + F(z, y) \tag{1.11}$$

where

$$F(z, y) = a_0 y''' + (c_1 y + c_0)y' + d_0 y'^2 + (e_2 y^2 + e_1 y + e_0)y' + f_4 y^4 + f_3 y^3 + f_2 y^2 + f_1 y + f_0 \tag{1.12}$$

and all the coefficients a, b, c, d, e, f with or without subscripts are assumed to be analytic functions of z.

In addition to their mathematically rich internal structure and frequent appearance in many physical problems, Painlevé equations play an important role for the completely integrable partial differential equations (PDEs). Ablowitz *et al* [10] demonstrated a close connection between completely integrable PDEs and Painlevé equations. They formulated the Painlevé conjecture or Painlevé ODE test. This conjecture provides a necessary condition to test whether a given PDE might be completely integrable. Weiss *et al* [11] introduced the Painlevé property for PDEs or the Painlevé PDE test as a method of applying the Painlevé ODE test directly to a given PDE without having to reduce it to an ODE.

Recently, Kudryashov [12] and Clarkson *et al* [13] obtained the higher-order Painlevé-type equations, the first and second Painlevé-hierarchy, by similarity reduction from the Korteweg-de Vries (KdV) and the modified Korteweg-de Vries (mKdV) hierarchies, respectively. The procedure used in [12, 13] can be summarized as follows: the KdV hierarchy can be written as

$$u_t + \frac{\partial}{\partial x}\mathcal{L}_n(u) = 0 \qquad n = 0, 1, 2 \tag{1.13}$$

where $\mathcal{L}_n$ satisfies the Lenard recursion relation

$$\frac{\partial}{\partial x}\mathcal{L}_{n+1} = \left(\frac{\partial^3}{\partial x^3} + 2u\frac{\partial}{\partial x} + u_x\right)\mathcal{L}_n \qquad n = 1, 2, 3 \tag{1.14}$$

beginning with $\mathcal{L}_0(u) = 1$, $\mathcal{L}_1(u) = u$. The KdV equation has the similarity reduction

$$u(x, t) = v(z) - \lambda t \qquad z = x + 3\lambda t^2 \tag{1.15}$$

with λ the arbitrary constant, where $v(z)$ is solvable in terms of the first Painlevé equation. By using the similarity reduction of KdV, one can obtain the first Painlevé hierarchy

$$\mathcal{P}_{n+1}(v) - \lambda z = 0 \qquad n = 0, 1, 2, 3 \tag{1.16}$$

where $\mathcal{P}_n$ satisfies the recursion relation

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{P}_{n+1} = \left(\frac{\mathrm{d}^3}{\mathrm{d}z^3} + v\frac{\mathrm{d}}{\mathrm{d}z} + v'\right)\mathcal{P}_n \qquad n = 1, 2, 3 \tag{1.17}$$

starting with $\mathcal{P}_0(v) = 1$ and $\mathcal{P}_1(v) = v$. Note that, for $n = 1$, equation (1.16) gives the first Painlevé equation

$$v'' = 6v^2 + z \tag{1.18}$$

and for $n = 2$

$$v^{(4)} + 5vv'' + \tfrac{5}{2}v'^2 + \tfrac{5}{2}v^3 + k_1 v - k_2 z = 0 \tag{1.19}$$

where k_i are arbitrary constants. Therefore, by using the operator $\mathcal{P}_n$, one can obtain the Painlevé-type equations of order $2n$ starting from the first Painlevé equation. In [12], the relation between the first and second Painlevé hierarchy was also examined.

In this paper the first Painlevé hierarchy is investigated by using the Painlevé ODE test, singular point analysis. It is possible to obtain a Painlevé-type equation of any order, as well as the known ones, starting from the first Painlevé equation. Singular point analysis is an algorithm introduced by Ablowitz *et al* [10] to test whether a given ODE satisfies the necessary conditions to be of Painlevé type.

The procedure to obtain higher-order Painlevé-type equations starting from the first Painlevé equation may be summarized as follows.

(I) Take an nth-order Painlevé-type differential equation

$$y^{(n)} = F(z, y, y', \ldots, y^{(n-1)}) \tag{1.20}$$

where F is analytic in z and rational in its other arguments. If $y \sim y_0(z - z_0)^\alpha$ as $z \to z_0$, then α is a negative integer for certain values of y_0. Moreover, the highest derivative term is one of the dominant terms. Then the dominant terms are of order $\alpha - n$. There are n resonances $r_0 = -1, r_1, r_2, \ldots, r_{n-1}$, for all $a = 1, 2, \ldots, (n-1)$ being non-negative real distinct integers such that $Q(r_j) = 0$, $j = 0, 1, 2, \ldots, (n-1)$. The compatibility conditions, for the simplified equation that retains only dominant terms of (1.20) are identically satisfied. Differentiating the simplified equation with respect to z yields

$$y^{(n+1)} = G(z, y, y', \ldots, y^{(n)}) \tag{1.21}$$

where G contains the terms of order $\alpha - n - 1$, and the resonances of (1.21) are the roots of $Q(r_j)(\alpha + r - n) = 0$. Hence, equation (1.21) has a resonance $r_n = n - \alpha$ in addition to the resonances of (1.20). Equation (1.21) passes the Painlevé test provided that r_n is a positive integer and $r_n \neq r_i$, $i = 1, 2, \ldots, (n-1)$ and is a positive integer. Moreover, the compatibility conditions are identically satisfied, that is $z_0, y_{r_1}, \ldots, y_{r_n}$ are arbitrary.

(II) Add the dominant terms which are not contained in G. Then the resonances of the new equation are the zeros of a polynomial $\tilde{Q}(r)$ of order $n+1$. Find the coefficients of $\tilde{Q}(r)$ such that there is at least one principal Painlevé branch. That is, all $n+1$ resonances (except $r_0 = -1$) are real positive distinct integers for at least one possible choice of (α, y_0). The other possible choices of (α, y_0) may give the secondary Painlevé branch, that is all the resonances are distinct integers.

(III) Add the non-dominant terms which are the terms of weight less than $\alpha - n - 1$, with analytic coefficients of z. Find the coefficients of the non-dominant terms by using the compatibility conditions.

The Painlevé test was improved in such a way so that negative resonances can be treated [14]. In this paper, we will consider only the 'principal branch' that is, all the resonances r_i (except $r_0 = -1$) are positive real distinct integers and the number of resonances is equal to the order of the differential equation for a possible choice of (α, y_0). Then, the compatibility conditions give a full set of arbitrary integration constants. The other possible choices of (α, y_0) may give a 'secondary branch' which possess several distinct negative integer resonances. Negative but distinct integer resonances give no conditions which contradict integrability [15]. In this paper, we start with the first Painlevé equation and obtain the third-, fourth-, fifth- and sixth-order equations of Painlevé type. A similar procedure can be used by starting from P_{II}, $P_{III}, \ldots, P_{VI}$ to obtain the higher-order equations. These results will be published elsewhere.

2. Third-order equations: $P_I^{(3)}$

The first Painlevé equation, P_I is

$$y'' = 6y^2 + z. \tag{2.1}$$

The Painlevé test gives that there is only one branch and

$$(\alpha, y_0) = (-2, 1) \qquad Q(r) = r^2 - 5r - 6. \tag{2.2}$$

The dominant terms are y'' and y^2 which are of order -4 as $z \to z_0$. Taking the derivative of the simplified equation gives

$$y''' = ayy' \tag{2.3}$$

where a is a constant which can be introduced by replacing y with λy, such that $12\lambda = a$. For equation (2.3), $(\alpha, y_0) = (-2, 12/a)$. No more polynomial-type terms of weight -5 with constant coefficients can be added to (2.3). The resonances of (2.3) are the zeros of

$$\tilde{Q}(r) = Q(r)(r-4). \tag{2.4}$$

Hence, the resonances are $(r_0, r_1, r_2) = (-1, 4, 6)$. The next step is to add the terms of weight less than -5 with analytic coefficients of z. That is,

$$y''' = ayy' + A(z)y'' + B(z)y^2 + C(z)y' + D(z)y + E(z). \tag{2.5}$$

The linear transformation

$$y(z) = \mu(z)u(t) + \nu(z) \qquad t = \rho(z) \tag{2.6}$$

where μ, ν and ρ are analytic functions of z, which preserves the Painlevé property. By using the transformation (2.6), one can set

$$aA + 2B = 0 \qquad C = 0 \qquad a = 12. \tag{2.7}$$

Then, substituting

$$y = y_0(z - z_0)^{-2} + \sum_{j=1}^{6} y_j(z - z_0)^{j-2} \tag{2.8}$$

into equation (2.6) gives that

$$y_0 = 1 \quad y_1 = 0 \quad y_2 = 0 \quad y_3 = D(z_0)/12. \tag{2.9}$$

The recursion relation for $j = 4$ implies that, if $y_4 =$ arbitrary, then

$$D' - AD = 0 \tag{2.10}$$

and for $j = 5$

$$y_5 = -\tfrac{1}{72}(12E_0 + 20B_0y_4 + 12D_2 + 2B_1D_0) \tag{2.11}$$

where B_k, $k = 0, 1, 2, \ldots$ and similarly D_k, E_k denote the coefficient of the kth-order term of Taylor series expansion of the appropriate function about $z = z_0$. The compatibility condition at the resonance $r = 6$ implies that

$$A' + A^2 = 0 \tag{2.12a}$$

$$-6(AE + E') - D(D - AA') + 3DA'' - 3AD'' - D''' = 0 \tag{2.12b}$$

if y_6 is arbitrary. According to (2.12a), there are two cases that should be considered separately.

(I) $A(z) = 0$. Equations (2.7), (2.10) and (2.12b) imply that $B = 0$, $D = c_1 =$ constant, $E(z) = -(c_1^2/6)z + c_2$, $c_2 =$ constant. Then the canonical form of a third-order Painlevé-type equation is

$$y''' = 12yy' + c_1y - \tfrac{1}{6}c_1^2z + c_2. \tag{2.13}$$

If $c_1 = c_2 = 0$, then (2.13) has the first integral

$$y'' = 6y^2 + k \qquad k = \text{constant} \tag{2.14}$$

which has the solution in terms of the elliptic functions. If $c_1 \neq 0$, then replace $z + c_2/k^2$ by z where $k = -c_1/6$, and then replace y by βy and z by γz such that $\gamma^2\beta = 1$ and $k\gamma^3 = -1$ in (2.13). It then takes the form of

$$y''' = 12yy' + 6y - 6z. \tag{2.15}$$

If one lets $y = u'$, integrates with respect to z once and replaces u by $u - c/6$ to eliminate the integration constant c, then (2.15) gives

$$u''' = 6u'^2 + 6u - 3z^2. \tag{2.16}$$

Equation (2.16) was also given by Chazy and Bureau [8, 9].

(II) $A(z) = 1/(z - c_1)$. Equations (2.7), (2.10) and (2.12b) give

$$B = -\frac{6}{z - c_1} \qquad D = c_2(z - c_1) \qquad E = -\frac{1}{24}c_2^2(z - c_1)^3 + \frac{c_3}{z - c_1} \tag{2.17}$$

where c_i, $i = 1, 2, 3$, are constants. Then the canonical form after replacing $z - c_1$ by z is

$$y''' = 12yy' + \frac{1}{z}(y'' - 6y^2) + c_2zy + \frac{c_3}{z} - \frac{c_2^2}{24}z^3. \tag{2.18}$$

Equation (2.18) was also considered in [9]. Replacing z by γz and y by βy, such that $\gamma^2\beta = 1$ and $c_2\gamma^4 = 12$ reduces equation (2.18) to

$$y''' = 12yy' + \frac{1}{z}(y'' - 6y^2 - k) + 12zy - 6z^3 \tag{2.19}$$

where k is an arbitrary constant. Integrating (2.19) once yields

$$\left(u'' - 6u^2 - \frac{k_1}{4}\right)^2 = z^2\left(u'^2 - 4u^3 - \frac{k_1}{2}u\right) \tag{2.20}$$

where $k_1 = -(k + 72)/3$ and $u = y - z^2/12$. There exists a one-to-one correspondence between $u(z)$ and the solution of the fourth Painlevé equation [7].

3. Fourth-order equations: $\mathbf{P_I^{(4)}}$

Differentiating (2.3) with respect to z gives the terms $y^{(4)}$, y'^2, yy'', all of which are of order -6 for $\alpha = -2$ and as $z \to z_0$. Adding the term y^3 which is also of order -6, gives the following simplified equation:

$$y^{(4)} = a_1y'^2 + a_2yy'' + a_3y^3 \tag{3.1}$$

where a_i, $i = 1, 2, 3$ are constants. Substituting

$$y = y_0(z - z_0)^{-2} + \beta y_1(z - z_0)^{r-2} \tag{3.2}$$

into the above equation gives the following equations for resonance r and for y_0, respectively:

$$Q(r) = (r + 1)[r^3 - 15r^2 + (86 - a_2y_0)r + 2(2a_1y_0 + 3a_2y_0 - 120)] = 0 \tag{3.3a}$$

$$a_3y_0^2 + 2(2a_1 + 3a_2)y_0 - 120 = 0. \tag{3.3b}$$

Equation (3.3*b*) implies that, in general, there are two branches of Painlevé expansion, if $a_3 \neq 0$. Now, one should determine y_{0j}, $j = 1, 2$ and a_i such that at least one of the branches is the principal branch. That is, all the resonances (except $r_0 = -1$ which is common for both branches) are distinct positive integers for one of $(-2, y_{0j})$, $j = 1, 2$. Negative but distinct resonances for the secondary branch may be allowed, since they give no conditions which contradict the Painlevé property.

If y_{01}, y_{02} are the roots of (3.3*b*), by setting

$$P(y_{0j}) = -2[(2a_1 + 3a_2)y_{0j} - 120] \qquad j = 1, 2 \tag{3.4}$$

and if (r_1, r_2, r_3), $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)$ are the resonances corresponding to y_{01} and y_{02} respectively, then one can have

$$\prod_{i=1}^{3} r_i = P(y_{01}) = p \qquad \prod_{i=1}^{3} \tilde{r}_i = P(y_{02}) = q \tag{3.5}$$

where p, q are integers and are such that, at least one of them is positive. Equation (3.3*b*) gives

$$y_{01} + y_{02} = -\frac{2}{a_3}(2a_1 + 3a_2) \qquad y_{01}y_{02} = -\frac{120}{a_3}. \tag{3.6}$$

Then equation (3.4) can be written as

$$P(y_{01}) = 120\left(1 - \frac{y_{01}}{y_{02}}\right) \tag{3.7a}$$

$$P(y_{02}) = 120\left(1 - \frac{y_{02}}{y_{01}}\right). \tag{3.7b}$$

Then, for $pq \neq 0$, p, q satisfy the following Diophantine equation:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{120}. \tag{3.8}$$

Now, one should determine all integer solutions of the Diophantine equation under certain conditions. Equation (3.3a) implies that $\sum_{i=1}^{3} r_i = \sum_{i=1}^{3} \tilde{r}_i = 15$. Let (r_1, r_2, r_3) be the distinct positive integers, then $r_1 + r_2 + r_3 = 15$ implies that there are 12 possible choices of (r_1, r_2, r_3). Then (3.8) has negative integer solutions q for each of the possible values of p except $p = 120$. The case $p = 120$ which corresponds to $(r_1, r_2, r_3) = (4, 5, 6)$ will be considered later. The equations (3.6), (3.7a) and $\sum_{i \neq j} r_i r_j = 86 - a_2 y_{01}$ determine y_{01}, y_{02}, a_1, a_3 in terms of a_2. Hence, all the coefficients of (3.3a) are determined such that its roots (r_1, r_2, r_3) corresponding to y_{01} are positive distinct integers, and the roots $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)$ corresponding to y_{02} are distinct integers such that $\prod_{i=1}^{3} \tilde{r}_i = q < 0$. Then, it should be checked whether the resonances $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)$ are distinct integers (i.e. existence of the secondary branch). There are four out of 11 cases such that (r_1, r_2, r_3) corresponding to y_{01} are positive distinct integers and $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)$ corresponding to y_{02} are distinct integers. These cases are as follows.

Case 1:

$$\begin{aligned} &y_{01} = \frac{30}{a_2} \qquad (r_1, r_2, r_3) = (2, 3, 10) \\ &y_{02} = \frac{60}{a_2} \qquad (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3) = (-2, 5, 12) \\ &a_1 = 0 \qquad a_3 = -\tfrac{1}{15} a_2^2. \end{aligned} \tag{3.9}$$

Case 2:

$$\begin{aligned} &y_{01} = \frac{20}{a_2} \qquad (r_1, r_2, r_3) = (2, 5, 8) \\ &y_{02} = \frac{60}{a_2} \qquad (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3) = (-3, 8, 10) \\ &a_1 = \tfrac{1}{2} a_2 \qquad a_3 = -\tfrac{1}{10} a_2^2. \end{aligned} \tag{3.10}$$

Case 3:

$$\begin{aligned} &y_{01} = \frac{18}{a_2} \qquad (r_1, r_2, r_3) = (3, 4, 8) \\ &y_{02} = \frac{90}{a_2} \qquad (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3) = (-5, 8, 12) \\ &a_1 = \tfrac{1}{2} a_2 \qquad a_3 = -\tfrac{2}{27} a_2^2. \end{aligned} \tag{3.11}$$

Case 4:

$$\begin{aligned} &y_{01} = \frac{15}{a_2} \qquad (r_1, r_2, r_3) = (3, 5, 7) \\ &y_{02} = \frac{120}{a_2} \qquad (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3) = (-7, 10, 12) \\ &a_1 = \tfrac{3}{4} a_2 \qquad a_3 = -\tfrac{1}{15} a_2^2. \end{aligned} \tag{3.12}$$

For each case the compatibility conditions are identically satisfied. To find the canonical form of the fourth-order equations of Painlevé type, one should add non-dominant terms with the coefficients which are analytic functions of z. That is, one should consider the following equation:

$$\begin{aligned} y^{(4)} = {} & a_1 y'^2 + a_2 y y'' + a_3 y^3 + A(z) y''' + B(z) y y' + C(z) y'' + D(z) y^2 + E(z) y' \\ & + F(z) y + G(z). \end{aligned} \tag{3.13}$$

The coefficients $A, \ldots, G$ can be determined by using the compatibility conditions.

Case 1. By using the linear transformation (2.6), one can set

$$2a_2A + 5B = 0 \qquad C = 0 \qquad a_2 = 30. \tag{3.14}$$

Substituting

$$y = y_{01}(z - z_0)^{-2} + \sum_{j=1}^{r_3} y_j(z - z_0)^{j-2} \tag{3.15}$$

into equation (3.13) gives the recursion relation for y_j. The recursion relation yields $y_1 = 0$ for $j = 1$ and for $j = r_1 = 2$, $D = 0$ if y_2 is arbitrary. If y_3 is arbitrary, then $B = E = 0$ and then the first equation of (3.14) implies that $A = 0$. The recursion relation for $j = r_3 = 10$ implies that $F = c_1 =$ constant and $G = c_2 =$ constant if y_{10} is arbitrary. Therefore, the canonical form is

$$y^{(4)} = 30yy'' - 60y^3 + c_1y + c_2. \tag{3.16}$$

For $c_1 = 0$, replacing y by $-y$ yields

$$y^{(4)} = -30yy'' - 60y^3 + c_2 \tag{3.17}$$

where $y(z)$ is the stationary solution of Caudrey–Dodd–Gibbon equation [16].

Case 2. Linear transformation (2.6) allows one to set

$$3a_2A + 5B = 0 \qquad C = 0 \qquad a_2 = 20. \tag{3.18}$$

Then, the compatibility conditions imply that $D = 0$ for $j = 2$, $B = E = 0$, $F(z) = c_1 =$ constant for $j = 5$ and $G = c_2z + c_3$, c_2 and c_3 are constant, for $j = 8$. Then the canonical form for this case is

$$y^{(4)} = 10(2yy'' + y'^2 - 4y^3) + c_1y + c_2z + c_3. \tag{3.19}$$

One can always choose $c_3 = 0$ by replacing $z + c_3/c_2$ by z. Replacing y by $-y/4$ in (3.19) gives

$$y^{(4)} + 5yy'' + \tfrac{5}{2}y'^2 + \tfrac{5}{2}y^3 + k_1y + k_2z = 0 \tag{3.20}$$

where $k_i =$ constant. Equation (3.20) was also introduced by Kudryashov [12].

Case 3. By using the linear transformation (2.6), one can set

$$2a_2A + 3B = 0 \qquad a_2C + 3D = 0 \qquad a_2 = 18. \tag{3.21}$$

Then, the compatibility conditions imply that $B = E = F = 0$ and $C = c_1$, $D = -6c_1$, $G = c_2z + c_3$, where c_i, $i = 1, 2, 3$ are constants. Therefore, the canonical form of the fourth-order Painlevé-type equation for this case is

$$y^{(4)} = 18yy'' + 9y'^2 - 24y^3 + c_1y'' - 6c_1y^2 + c_2z + c_3. \tag{3.22}$$

For $c_2 \neq 0$, replacing $z + c_3/c_2$ by z and then replacing z by γz and y by βy such that $\beta\gamma^2 = 1$, $c_2\gamma^7 = 1$ reduces (3.22) into the following form:

$$y^{(4)} = 18yy'' + 9y'^2 - 24y^3 + k_1y'' - 6k_1y^2 + z \tag{3.23}$$

where $k_1 = c_1\gamma^2$.

Case 4. Linear transformation (2.6) allows one to set

$$4a_2A + 5B = 0 \qquad D = 0 \qquad a_2 = 15. \tag{3.24}$$

Then the compatibility conditions at the resonances $j = 3, 5, 7$ imply that, if y_3, y_5, y_7 are arbitrary then $B = C = E = 0$ and $F = c_1 =$ constant, $G = c_2 =$ constant. Therefore, the canonical form is

$$y^{(4)} = 15yy'' + \tfrac{45}{4}y'^2 - 15y^3 + c_1y + c_2. \tag{3.25}$$

If one sets $y = -2u$ then (3.25) takes the form of

$$u^{(4)} + 30uu'' + \tfrac{45}{2}u'^2 + 60u^3 + k_1u + k_2 = 0 \tag{3.26}$$

where $k_1 = -c_1$, $k_2 = c_2/2$. $u(z)$ is the stationary solution of the Kuperschmidt equation [16] for $k_1 = 0$.

If $a_3 = 0$, equation (3.3) reduces to

$$Q(r) = (r + 1)[r^3 - 15r^2 + (86 - a_2y_0)r - 120] = 0 \tag{3.27a}$$

$$(2a_1 + 3a_2)y_0 - 60 = 0 \tag{3.27b}$$

and hence, there is only one Painlevé branch which has to be the principal branch. (3.27a) implies that $r_0 = -1$ and $\sum_{i=1}^{3} r_i = 15$ which gives 12 possible positive distinct integers (r_1, r_2, r_3). However, $\prod_{i=1}^{3} r_i = 120$ implies that $(r_1, r_2, r_3) = (4, 5, 6)$ is the only possible choice of the resonances. Equation (3.27b) and $\sum_{i\neq j} r_ir_j = 86 - a_2y_0$ imply that $a_1 = a_2$. Then, the simplified equation is

$$y^{(4)} = a_1(yy'' + y'^2). \tag{3.28}$$

Adding the non-dominant terms with the analytic coefficients of z gives

$$y^{(4)} = a_1(yy'' + y'^2) + A(z)y''' + B(z)yy' + C(z)y'' + D(z)y^2 + E(z)y' + F(z)y + G(z). \tag{3.29}$$

One can always set

$$a_2A + B = 0 \qquad C = 0 \qquad a_2 = 12 \tag{3.30}$$

by using the linear transformation (2.6). The compatibility conditions at the resonances $r = 4, 5, 6$ imply that y_4, y_5, y_6 are arbitrary and $B = D = 0$ and

$$E = \frac{c_1}{2}z + c_2 \qquad F = c_1 \qquad G = -\frac{1}{6}\left(\frac{c_1}{2}z + c_2\right)^2 \tag{3.31}$$

where c_1, c_2 are constants. Hence, the canonical form is

$$y^{(4)} = 12(yy'' + y'^2) + \left(\frac{c_1}{2}z + c_2\right)y' + c_1y - \frac{1}{6}\left(\frac{c_1}{2}z + c_2\right)^2. \tag{3.32}$$

If $c_1 = 0$, then integrating (3.32) once gives equation (2.15). If $c_1 \neq 0$, letting $c_1 = -12k_1$, $c_2 = -6k_2$ first, replacing $z + k_2/k_1$ by z, and then replacing z by γz, y by βy, such that $\beta\gamma^2 = 1$, $k_1\gamma^4 = 1$ then equation (3.32) takes the form of

$$y^{(4)} = 12(yy')' - 6zy' - 12y - 6z^2. \tag{3.33}$$

If one lets $y = -u'$ and integrates the resulting equation once then (3.33) yields

$$u^{(4)} + 12u'u'' = 6zu' + 6u + 2z^3 - k \tag{3.34}$$

after replacing u by βu and z by γz such that $\beta\gamma = -1$, $\gamma^4 = -1$. Equation (3.34) was also obtained by Bureau [9] and belongs to the second Painlevé equation.

4. Fifth-order equations: $\mathbf{P}_{\mathbf{I}}^{(5)}$

Differentiating (3.1) with respect to z gives the terms $y^{(5)}$, yy''', $y'y''$, y^2y' which are all the dominant terms for $\alpha = -2$ and $z \to z_0$. Therefore, the simplified equation is

$$y^{(5)} = a_1 yy''' + a_2 y'y'' + a_3 y^2 y' \tag{4.1}$$

where a_i, $i = 1, 2, 3$ are constants. Substituting (3.2) into (4.1) gives the following equations for the resonance r and y_0:

$$(r+1)\{r^4 - 21r^3 + (176 - a_1 y_0)r^2 + [2(5a_1 + a_2)y_0 - 378]r$$
$$+[1800 - 18(2a_1 + a_2)y_0 - a_3 y_0^2]\} = 0 \qquad (4.2a)$$

$$a_3 y_0^2 + 6(2a_1 + a_2)y_0 - 360 = 0. \qquad (4.2b)$$

Equation (4.2*a*) implies that one of the resonances, $r_0 = -1$, corresponds to the arbitrariness of z_0. (4.2*b*) implies the existence of two Painlevé branches corresponding to $(-2, y_{0i})$, $i = 1, 2$. Let (r_1, r_2, r_3, r_4) and $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4)$ be the resonances corresponding to y_{01} and y_{02}, respectively. Setting,

$$P(y_{0j}) = 1800 - 18(2a_1 + a_2)y_{0j} - a_3 y_{0j}^2 \qquad j = 1, 2 \tag{4.3}$$

then, (4.2*a*) implies that

$$\prod_{i=1}^{4} r_i = P(y_{01}) = p \qquad \prod_{i=1}^{4} \tilde{r}_i = P(y_{02}) = q \tag{4.4}$$

where p, q are integers such that at least one of them is positive, to have the principal branch. From equation (4.2*b*), one can have

$$a_3 = -\frac{360}{y_{01}y_{02}} \qquad 2a_1 + a_2 = \frac{60}{y_{01}y_{02}}(y_{01} + y_{02}). \tag{4.5}$$

By using the above equation, (4.3) yields the following Diophantine equation, if $pq \neq 0$:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{720}. \tag{4.6}$$

Now, one should determine all possible integer solutions (p, q) of (4.6). (4.2*a*) implies that $\sum_{i=1}^{4} r_i = 21$. Then, there are 27 possible cases for (r_1, r_2, r_3, r_4) (i.e. 27 possible values of p) such that r_i are positive distinct integers. The Diophantine equation implies that there are 12 out of 27 cases such that both $p > 0, q < 0$ are integers. By using the equations

$$\sum_{i \neq j} r_i r_j = 176 - a_1 y_{01} \qquad \sum_{i \neq j \neq k} r_i r_j r_k = -2[(5a_1 + a_2)y_{01} - 378] \tag{4.7}$$

and (4.5), y_{01}, y_{02}, a_2, a_3 can be obtained in terms of a_1 for each 12 possible integer values of (p, q). However, there are only four out of 12 cases such that the resonances $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4)$ corresponding to y_{02} are distinct integers. These cases and the corresponding simplified equations are as follows.

Case 1:

$$\begin{aligned} &y_{01} = \frac{30}{a_1} \qquad (r_1, r_2, r_3, r_4) = (2, 3, 6, 10) \\ &y_{02} = \frac{60}{a_1} \qquad (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4) = (-2, 5, 6, 12) \\ &a_2 = a_1 \qquad a_3 = -\tfrac{1}{5}a_1^2 \qquad y^{(5)} = a_1(yy''' + y'y'' - \tfrac{1}{5}a_1 y^2 y'). \end{aligned} \tag{4.8}$$

Case 2:

$$\begin{aligned} &y_{01} = \frac{15}{a_1} \qquad (r_1, r_2, r_3, r_4) = (3, 5, 6, 7) \\ &y_{02} = \frac{120}{a_1} \qquad (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4) = (-7, 6, 10, 12) \\ &a_2 = \tfrac{5}{2}a_1 \qquad a_3 = -\tfrac{1}{5}a_1^2 \qquad y^{(5)} = a_1(yy''' + \tfrac{5}{2}y'y'' - \tfrac{1}{5}a_1y^2y'). \end{aligned} \tag{4.9}$$

Case 3:

$$\begin{aligned} &y_{01} = \frac{18}{a_1} \qquad (r_1, r_2, r_3, r_4) = (3, 4, 6, 8) \\ &y_{02} = \frac{90}{a_1} \qquad (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4) = (-5, 6, 8, 12) \\ &a_2 = 2a_1 \qquad a_3 = -\tfrac{2}{9}a_1^2 \qquad y^{(5)} = a_1(yy''' + 2y'y'' - \tfrac{2}{9}a_1y^2y'). \end{aligned} \tag{4.10}$$

Case 4:

$$\begin{aligned} &y_{01} = \frac{20}{a_1} \qquad (r_1, r_2, r_3, r_4) = (2, 5, 6, 8) \\ &y_{02} = \frac{60}{a_1} \qquad (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4) = (-3, 6, 8, 10) \\ &a_2 = 2a_1 \qquad a_3 = -\tfrac{3}{10}a_1^2 \qquad y^{(5)} = a_1(yy''' + 2y'y'' - \tfrac{3}{10}a_1y^2y'). \end{aligned} \tag{4.11}$$

The compatibility conditions for all four cases are identically satisfied.

To obtain the canonical form of the fifth-order equation of Painlevé type, one should add the non-dominant terms of weight <7 for $\alpha = -2$ with analytic coefficients of z. Therefore, the general form is

$$\begin{aligned} y^{(5)} = {} & a_1yy''' + a_2y'y'' + a_3y^2y' + A(z)y^{(4)} + B(z)y''' + C(z)yy'' + D(z)y'' + E(z)y'^2 \\ & + F(z)yy' + G(z)y' + H(z)y^3 + J(z)y^2 + K(z)y + L(z). \end{aligned} \tag{4.12}$$

The coefficients $A(z), \ldots, L(z)$ can be determined by using the compatibility conditions. Substituting

$$y = y_{01}(z - z_0)^{-2} + \sum_{j=1}^{r_4} y_j(z - z_0)^{j-2} \tag{4.13}$$

into (4.12) gives the recursion relation for y_j. The recursion relations for $j = r_1, r_2, r_3, r_4$ give the compatibility conditions if $y_{r_1}, y_{r_2}, y_{r_3}, y_{r_4}$ are arbitrary.

Case 1. By using the linear transformation (2.6), one can set

$$2a_1^2A + 3a_1C + 2a_1E + 15H = 0 \qquad F = 0 \quad a_1 = 30 \tag{4.14}$$

then, $y_{01} = 1$ and $y_1 = 0$. The compatibility conditions at $j = 2, 3, 6, 10$ imply that all the coefficients are zero except

$$G = c_1z + c_2 \qquad K = 2c_1 \tag{4.15}$$

where c_1, c_2 are constants. Then the canonical form for this case is

$$y^{(5)} = 30(yy''' + y'y'' - 6y^2y') + (c_1z + c_2)y' + 2c_1y. \tag{4.16}$$

If $c_1 \neq 0$, replacing $z + c_2/c_1$ by z and then replacing z by γz and y by βy such that $\gamma^2\beta = 1$, $c_1\gamma^5 = 1$ in (4.16) gives

$$y^{(5)} = 30(yy''' + y'y'' - 6y^2y') + zy' + 2y. \tag{4.17}$$

Case 2. One can always choose

$$8a_1^2A + 6a_1C + 4a_1E + 15H = 0 \qquad 4a_1B + 5F = 0 \quad a_1 = 15 \tag{4.18}$$

by using the linear transformation (2.6). Then $y_{01} = 1$, $y_1 = y_2 = 0$. The compatibility conditions at $j = 3, 5, 6, 7$ imply that all the coefficients are zero except

$$G = c_1z + c_2 \qquad K = 2c_1 \tag{4.19}$$

where c_1, c_2 are constants. Then the canonical form for this case is

$$y^{(5)} = 15(yy''' + \tfrac{5}{2}y'y'' - 3y^2y') + (c_1z + c_2)y' + 2c_1y. \tag{4.20}$$

If $c_1 \neq 0$, replacing $z + c_2/c_1$ by z and then replacing z by γz and y by βy such that $\gamma^2\beta = 1$, $c_1\gamma^5 = 1$ in (4.20) gives

$$y^{(5)} = 15(yy''' + \tfrac{5}{2}y'y'' - 3y^2y') + zy' + 2y. \tag{4.21}$$

Case 3. By using the transformation (2.6) one can set $y_{01} = 1$, $y_1 = y_2 = 0$. That is,

$$10a_1^2A + 9a_1C + 6a_1E + 27H = 0 \qquad 6a_1B + 9F = 0 \quad a_1 = 18. \tag{4.22}$$

The compatibility conditions at $j = 3, 4, 6, 8$ give

$$6D + J = 0 \tag{4.23}$$

$$-6C + 4E - 3H = 0 \qquad G = 0 \tag{4.24}$$

$$24F' - 48J - FH = 0 \qquad -24K' + HK = 0 \tag{4.25}$$

and

$$8E + 3H = 0 \qquad 24H' + H^2 = 0 \qquad 24J' + HJ = 0 \tag{4.26}$$

respectively. The second equation of (4.26) implies that there are two cases that should be considered separately.

(a) $H(z) = 0$. Equations (4.22)–(4.26) and the compatibility condition at $j = 8$ imply that all the coefficients are zero except

$$F = c_1 \qquad B = -\tfrac{1}{6}c_1 \qquad L = c_2 \tag{4.27}$$

where c_1, c_2 are constants. Then, the canonical form of the equation for this case is

$$y^{(5)} = 18(yy''' + 2y'y'' - 4y^2y') - \tfrac{1}{6}c_1y''' + c_1yy' + c_2. \tag{4.28}$$

(b) $H(Z) = 24/(z - c)$: For simplicity, let the constant $c = 0$. Then equations (4.22)–(4.26) and the compatibility condition at $j = 8$ implies that there are the two following distinct cases:

(i)

$$A = \frac{1}{z} \quad B = \frac{c_2}{6} \quad C = -\frac{18}{z} \quad D = -\frac{c_2}{6z} \quad E = -\frac{9}{z}$$
$$F = -2c_2 \quad J = \frac{c_2}{z} \quad K = 0, \quad L = \frac{c_1}{z} \tag{4.29}$$

where c_1, c_2 are constants. Then, the canonical form is

$$y^{(5)} = 18(yy''' + 2y'y'' - 4y^2y') + \frac{1}{z}y^{(4)} + \frac{c_2}{6}y''' - \frac{18}{z}yy''$$
$$-\frac{c_2}{6z}y'' - \frac{9}{z}y'^2 - 2c_2yy' + \frac{24}{z}y^3 + \frac{c_2}{z}y^2 + \frac{c_1}{z}. \tag{4.30}$$

When $c_2 = 0$; if one lets

$$u = y^{(4)} - 3(6yy'' + 3y'^2 - 8y^3) \tag{4.31}$$

then equation (4.30) can be written as

$$u' = \frac{1}{z}u + \frac{c_1}{z}. \tag{4.32}$$

Hence, (4.30) has the first integral

$$y^{(4)} = 3(6yy'' + 3y'^2 - 8y^3) + kz - c_1 \tag{4.33}$$

where k is an arbitrary constant. Equation (4.33) is nothing but equation (3.23) with $k_1 = 0$.

(ii) $D = G = J = 0$ and

$$\begin{aligned} &A = \frac{1}{z} \quad B = -\frac{c_3}{2}z \quad C = -\frac{18}{z} \quad E = -\frac{9}{z} \\ &F = 6c_3z \quad K = \frac{c_3^2}{2}z \quad L = -\frac{c_3^3}{36}z^2 + \frac{c_4}{z} \end{aligned} \tag{4.34}$$

where c_3, c_4 are constants. Then, the canonical form is

$$\begin{aligned} y^{(5)} = {} & 18(yy''' + 2y'y'' - 4y^2y') + \frac{1}{z}y^{(4)} - \frac{c_3}{2}zy''' - \frac{18}{z}yy'' \\ & -\frac{9}{z}y'^2 + 6c_3zyy' + \frac{24}{z}y^3 + \frac{c_3^2}{2}zy - \frac{c_3^3}{36}z^2 + \frac{c_4}{z}. \end{aligned} \tag{4.35}$$

When $c_3 = 0$, (4.35) has the same first integral as (4.33).

Case 4. By using the transformation one can set

$$3a_1^2A + 3a_1C + 2a_1E + 10H = 0 \qquad F = 0 \quad a_1 = 20. \tag{4.36}$$

The compatibility conditions at $j = 2, 5$ imply that $B = 0$ and $D = 0$, respectively. The compatibility conditions at $j = 6, 8$ imply that

$$4E + H = 0 \tag{4.37}$$

and

$$J = 0 \qquad -7C + 6E - 2H = 0 \qquad 40H' + H^2 = 0 \qquad 40K' + KH = 0 \tag{4.38}$$

respectively. Therefore, there are two cases that should be considered separately: (a) $H(z) = 0$ and (b) $H(z) = 40/z$ (for simplicity the integration constant is set to zero).

(a) $H(z) = 0$. Equations (4.36)–(4.38) imply that all the coefficients are zero except $G = c_1z + c_2$, $K = 2c_1$ and $L(z) = c_3$, where c_i are constants. Then, the canonical form is

$$y^{(5)} = 20(yy''' + 2y'y'' - 6y^2y') + (c_1z + c_2)y' + 2c_1y + c_3. \tag{4.39}$$

(b) $H(z) = 40/z$. Equations (4.36)–(4.38) and the compatibility conditions at $j = 5, 8$ imply that

$$\begin{aligned} &A = \frac{1}{z} \quad B = 0 \quad C = -\frac{20}{z} \quad D = 0 \quad E = -\frac{10}{z} \\ &F = 0 \quad G = -k_1 \quad J = 0 \quad K = \frac{k_1}{z} \quad L = \frac{k_2}{z} \end{aligned} \tag{4.40}$$

where k_1, k_2 are constants. Then, the canonical form is

$$y^{(5)} = 20(yy''' + 2y'y'' - 6y^2y') + \frac{1}{z}y^{(4)} - \frac{20}{z}yy'' - \frac{10}{z}y'^2 - k_1y' + \frac{40}{z}y^3 + \frac{k_1}{z}y + \frac{k_2}{z}. \tag{4.41}$$

When $k_1 = 0$, if one lets

$$u = y^{(4)} - 10(2yy'' + y'^2 - 4y^3) \tag{4.42}$$

then equation (4.41) can be written as

$$u' = \frac{1}{z}u + \frac{k_2}{z}. \tag{4.43}$$

Hence, the first integral of (4.41) is

$$y^{(4)} = 10(2yy'' + y'^2 - 4y^3) + k_3 z - k_2 \tag{4.44}$$

where k_3 is an arbitrary constant. Replacing y by $-y/4$ in (4.44) gives (3.20) with $k_1 = 0$.

5. Sixth-order equations: $\mathbf{P_I^{(6)}}$

Differentiating (4.1) with respect to z gives the terms $y^{(6)}$, $yy^{(4)}$, $y'y'''$, y''^2, y^2y'', yy'^2 all of which are of order -8 for $\alpha = -2$ as $z \to z_0$. Adding the term y^4 which is also of order -8 gives the following simplified equation:

$$y^{(6)} = a_1 yy^{(4)} + a_2 y'y''' + a_3 y''^2 + a_4 y^2 y'' + a_5 yy'^2 + a_6 y^4 \tag{5.1}$$

where a_i, $i = 1, 2, \ldots, 6$ are constants. Substituting (3.2) into (5.1) gives the following equations for the resonance r and y_0:

$$\begin{aligned}(r+1)\{r^5 - 28r^4 &+ (323 - a_1 y_0)r^3 + [(15a_1 + 2a_2)y_0 - 1988]r^2 \\ &-[a_4 y_0^2 + 2(43a_1 + 10a_2 + 6a_3)y_0 - 7092]r + 2[(2a_5 + 3a_4)y_0^2 \\ &+12(10a_1 + 4a_2 + 3a_3)y_0 - 7560]\} = 0\end{aligned} \tag{5.2a}$$

$$a_6 y_0^3 + 2(3a_4 + 2a_5)y_0^2 + 12(10a_1 + 4a_2 + 3a_3)y_0 - 5040 = 0. \tag{5.2b}$$

Equation (5.2a) implies that one of the resonances, $r_0 = -1$, corresponds to the arbitrariness of z_0. Two cases should now be considered separately: (a) $a_6 = 0$ and (b) $a_6 \neq 0$.

(a) $a_6 = 0$. There are two Painlevé branches corresponding to $(-2, y_{0j})$, $j = 1, 2$, where y_{0j} are the roots of

$$(3a_4 + 2a_5)y_0^2 + 6(10a_1 + 4a_2 + 3a_3)y_0 - 2520 = 0. \tag{5.3}$$

Then, one has

$$y_{01} + y_{02} = -\frac{6(10a_1 + 4a_2 + 3a_3)}{3a_4 + 2a_5} \qquad y_{01}y_{02} = -\frac{2520}{3a_4 + 2a_5}. \tag{5.4}$$

Let $r_1, r_2, \ldots, r_5$ and $\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_5$ be the roots (additional to $r_0 = -1$) of (5.2a) corresponding to y_{01} and y_{02}, respectively. Setting

$$P(y_{0j}) = -2[(2a_5 + 3a_4)y_{0j}^2 + 12(10a_1 + 4a_2 + 3a_3)y_{0j} - 7560] \qquad j = 1, 2 \tag{5.5}$$

then (5.2a) implies that

$$\prod_{i=1}^{5} r_i = P(y_{01}) = p \qquad \prod_{i=1}^{5} \tilde{r}_i = P(y_{02}) = q \tag{5.6}$$

and

$$\sum_{i=1}^{5} r_i = \sum_{i=1}^{5} \tilde{r}_i = 28 \tag{5.7}$$

where p, q are integers, and at least one of them is positive. Now, one should determine y_{0j}, $j = 1, 2$, and $a_i, i = 1, 2, \ldots, 5$ such that there is at least one principal branch. Let the branch corresponding to y_{01} be the principal branch, then $p > 0$. Equation (5.5) gives

$$P(y_{01}) = 5040\left(1 - \frac{y_{01}}{y_{02}}\right) = p \qquad P(y_{02}) = 5040\left(1 - \frac{y_{02}}{y_{01}}\right) = q \tag{5.8}$$

by using (5.4). Therefore, p, q satisfy the following Diophantine equation, if $pq \neq 0$:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{5040}. \tag{5.9}$$

Equation (5.7) implies that there are 57 possible cases of $(r_1, r_2, \ldots, r_5)$ such that r_i are positive distinct integers. The Diophantine equation has 27 integer solutions (p, q) such that $q < 0$. For each 27 cases of (p, q), y_{0j}, $j = 1, 2$, and a_i, $i = 2, \ldots, 5$ can be obtained from (5.4), (5.8) and

$$\begin{aligned}
&\sum_{i\neq j} r_i r_j = 323 - a_1 y_{01} \qquad \sum_{i\neq j\neq k} r_i r_j r_k = -(15a_1 + 2a_2)y_{01} + 1988] \\
&\sum_{i\neq j\neq k\neq l} r_i r_j r_k r_l = -a_4 y_{01}^2 - 2(43a_1 + 10a_2 + 6a_3)y_{01} + 7092
\end{aligned} \tag{5.10}$$

in terms of a_1. But there are only three out of 27 cases such that the resonances $(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_5)$ corresponding to y_{02} are distinct integers. These cases and the corresponding simplified equations are as follows.

Case 1:

$$\begin{aligned}
&y_{01} = \frac{20}{a_1} \qquad (r_1, r_2, r_3, r_4, r_5) = (2, 5, 6, 7, 8) \\
&y_{02} = \frac{60}{a_1} \qquad (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5) = (-3, 6, 7, 8, 10) \\
&a_2 = 3a_1 \qquad a_3 = 2a_1 \qquad a_4 = -\tfrac{3}{10}a_1^2 \qquad a_5 = -\tfrac{3}{5}a_1^2 \\
&y^{(6)} = a_1(yy^{(4)} + 3y'y''' + 2y''^2 - \tfrac{3}{10}a_1 y^2 y'' - \tfrac{3}{5}a_1 yy'^2).
\end{aligned} \tag{5.11}$$

Case 2:

$$\begin{aligned}
&y_{01} = \frac{18}{a_1} \qquad (r_1, r_2, r_3, r_4, r_5) = (3, 4, 6, 7, 8) \\
&y_{02} = \frac{90}{a_1} \qquad (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5) = (-5, 6, 7, 8, 12) \\
&a_2 = 3a_1 \qquad a_3 = 2a_1 \qquad a_4 = -\tfrac{2}{9}a_1^2 \qquad a_5 = -\tfrac{4}{9}a_1^2 \\
&y^{(6)} = a_1(yy^{(4)} + 3y'y''' + 2y''^2 - \tfrac{2}{9}a_1 y^2 y'' - \tfrac{4}{9}a_1 yy'^2).
\end{aligned} \tag{5.12}$$

Case 3:

$$\begin{aligned}
&y_{01} = \frac{30}{a_1} \qquad (r_1, r_2, r_3, r_4, r_5) = (2, 3, 6, 7, 10) \\
&y_{02} = \frac{60}{a_1} \qquad (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5) = (-2, 5, 6, 7, 12) \\
&a_2 = 2a_1 \qquad a_3 = a_1 \qquad a_4 = -\tfrac{1}{5}a_1^2 \qquad a_5 = -\tfrac{2}{5}a_1^2 \\
&y^{(6)} = a_1(yy^{(4)} + 2y'y''' + y''^2 - \tfrac{1}{5}a_1 y^2 y'' - \tfrac{2}{5}a_1 yy'^2).
\end{aligned} \tag{5.13}$$

The compatibility conditions are identically satisfied for the first two cases but not for the third case. Therefore, the third case will not be considered.

To obtain the canonical form of the sixth-order Painlevé-type equation when $a_6 = 0$, one should add the non-dominant terms with analytic coefficients of z. That is,

$$\begin{aligned}
y^{(6)} = {} & a_1 yy^{(4)} + a_2 y'y''' + a_3 y''^2 + a_4 y^2 y'' + a_5 yy'^2 + A(z)y^{(5)} + B(z)y^{(4)} + C(z)yy''' \\
& + D(z)y''' + E(z)y'y'' + F(z)yy'' + G(z)y'' + H(z)y^2 y' + J(z)yy' \\
& + K(z)y'^2 + L(z)y' + M(z)y^3 + N(z)y^2 + P(z)y + R(z).
\end{aligned} \tag{5.14}$$

The coefficients $A(z), \ldots, R(z)$ can be determined by using the compatibility conditions at the resonances. Substituting

$$y = y_{01}(z - z_0)^{-2} + \sum_{j=1}^{r_5} y_j (z - z_0)^{j-2} \tag{5.15}$$

into (5.14) gives the recursion relation for y_j. Then, one can find $A, \ldots, R$ such that the recursion relations for $j = r_1, r_2, r_3, r_4, r_5$ are identically satisfied, and hence $y_{r_1}, y_{r_2}, y_{r_3}, y_{r_4}, y_{r_5}$ are arbitrary.

Case 1. By using the linear transformation (2.6), one can set

$$9a_1^2 A + 6a_1 C + 3a_1 E + 10H = 0 \qquad F = 0 \quad a_1 = 20 \tag{5.16}$$

then, $y_{01} = 1$ and $y_1 = 0$. The compatibility conditions at $j = 2, 5, 6, 7, 8$ imply that all the coefficients are zero except

$$G = c_1 z + c_2 \qquad L = 3c_1 \tag{5.17}$$

where c_1, c_2 are constants. Then the canonical form for this case is

$$y^{(6)} = 20(yy^{(4)} + 3y'y''' + 2y''^2 - 6y^2 y'' - 12yy'^2) + (c_1 z + c_2)y'' + 3c_1 y'. \tag{5.18}$$

If $c_1 \neq 0$, replacing $z + c_2/c_1$ by z and then replacing z by γz and y by βy such that $\gamma^2\beta = 1$, $c_1\gamma^5 = 1$ in (5.18) gives

$$y^{(6)} = 20(yy^{(4)} + 3y'y''' + 2y''^2 - 6y^2 y'' - 12yy'^2) + zy'' + 3y'. \tag{5.19}$$

Case 2. One can always choose $y_{01} = 1$, and $y_1 = y_2 = 0$ by choosing

$$10a_1^2 A + 6a_1 C + 3a_1 E + 9H = 0 \qquad 10a_1^2 B + 9a_1 F + 6a_1 K + 27M = 0 \qquad a_1 = 18. \tag{5.20}$$

Then, the recursion relation imply that if, y_3, y_4, y_6, y_7, and y_8 are arbitrary then $A = C = E = G = H = M = N = 0$ and

$$\begin{aligned} &B = -\frac{1}{12}(c_1 z + c_2) \qquad D = -\frac{1}{6}c_1 \qquad F = K = c_1 z + c_2 \qquad J = 2c_1 \\ &L = \frac{c_1}{72}(c_1 z + c_2) \qquad P = \frac{1}{36}c_1^2 \qquad R = -\frac{c_1^2}{2592}(c_1 z + c_2) \end{aligned} \tag{5.21}$$

where c_1, c_2 are arbitrary constants. Then the canonical form for this case is

$$\begin{aligned} y^{(6)} = {}& 18(yy^{(4)} + 3y'y''' + 2y''^2 - 4y^2 y'' - 8yy'^2) - \frac{1}{12}(c_1 z + c_2)y^{(4)} - \frac{c_1}{6}y''' \\ &+(c_1 z + c_2)yy'' + 2c_1 yy' + (c_1 z + c_2)y'^2 + \frac{c_1}{72}(c_1 z + c_2)y' + \frac{c_1^2}{36}y \\ &-\frac{c_1^2}{2592}(c_1 z + c_2). \end{aligned} \tag{5.22}$$

If $c_1 \neq 0$, replacing $z + c_2/c_1$ by z and then replacing z by γz and y by βy such that $\gamma^2\beta = 1$, $c_1\gamma^3 = 36$ in (5.22) gives

$$\begin{aligned} y^{(6)} = {}& 18(yy^{(4)} + 3y'y''' + 2y''^2 - 4y^2 y'' - 8yy'^2) - 3zy^{(4)} - 6y''' + 36z(yy'' + y'^2) \\ &+6(12yy' + 3zy' + 6y - 3z). \end{aligned} \tag{5.23}$$

(b) $a_6 \neq 0$: Equation (5.2b) implies that there are three Painlevé branches corresponding to $(-2, y_{0j})$, $j = 1, 2, 3$ where y_{0j} are the roots of (5.2b). (5.2b) implies that

$$\prod_{j=1}^{3} y_{0j} = \frac{5040}{a_6} \qquad \sum_{j=1}^{3} y_{0j} = -\frac{2(3a_4 + 2a_5)}{a_6} \qquad \sum_{i\neq j} y_{0i}\, y_{0j} = \frac{12}{a_6}(10a_1 + 4a_2 + 3a_3). \tag{5.24}$$

If the resonances (except $r_0 = -1$) are $r_i, \tilde{r}_i, \hat{r}_i$, $i = 1, 2, \ldots, 5$ corresponding to y_{01}, y_{02}, y_{03}, respectively, and if one sets

$$P(y_{0j}) = -2[(2a_5 + 3a_4)y_{0j}^2 + 12(10a_1 + 4a_2 + 3a_3)y_{0j} - 7560] \tag{5.25}$$

then, (5.2a) implies that

$$\prod_{i=1}^{5} r_i = P(y_{01}) \qquad \prod_{i=1}^{5} \tilde{r}_i = P(y_{02}) \qquad \prod_{i=1}^{5} \hat{r}_i = P(y_{03}) \tag{5.26}$$

and

$$\sum_{i=1}^{5} r_i = \sum_{i=1}^{5} \tilde{r}_i = \sum_{i=1}^{5} \hat{r}_i = 28. \tag{5.27}$$

The condition of $r_i, \tilde{r}_i, \hat{r}_i$ being integers and (5.25), (5.26) give

$$P(y_{01}) = p_1 \qquad P(y_{02}) = p_2 \qquad P(y_{03}) = p_3 \tag{5.28}$$

where p_1, p_2, p_3 are integers, and at least one is positive. Then equations (5.24) and (5.25) give

$$\begin{aligned} p_1 &= 5040\left(1 - \frac{y_{01}}{y_{02}}\right)\left(1 - \frac{y_{01}}{y_{03}}\right) \qquad p_2 = 5040\left(1 - \frac{y_{02}}{y_{01}}\right)\left(1 - \frac{y_{02}}{y_{03}}\right) \\ p_3 &= 5040\left(1 - \frac{y_{03}}{y_{01}}\right)\left(1 - \frac{y_{03}}{y_{02}}\right). \end{aligned} \tag{5.29}$$

By setting, $\kappa = y_{02} - y_{03}$, $\mu = y_{03} - y_{01}$, and $\nu = y_{01} - y_{02}$, (5.29) then yields

$$p_1 = -5040\frac{\mu\nu}{y_{02}y_{03}} \qquad p_2 = -5040\frac{\kappa\nu}{y_{01}y_{03}} \qquad p_3 = -5040\frac{\kappa\mu}{y_{01}y_{02}}. \tag{5.30}$$

Thus,

$$\sum_{i\neq j} p_i p_j = (5040)^2\kappa\mu\nu\left(\frac{\kappa}{y_{01}} + \frac{\mu}{y_{02}} + \frac{\nu}{y_{03}}\right). \tag{5.31}$$

But

$$\frac{\kappa}{y_{01}} + \frac{\mu}{y_{02}} + \frac{\nu}{y_{03}} = -\frac{\kappa\mu\nu}{y_{01}y_{02}y_{03}}. \tag{5.32}$$

Therefore,

$$\sum_{i\neq j} p_i p_j = -(5040)^2\frac{\kappa^2\mu^2\nu^2}{y_{01}^2 y_{02}^2 y_{03}^2} = \frac{1}{5040}p_1 p_2 p_3 \tag{5.33}$$

so that, p_i, $i = 1, 2, 3$, satisfy the following Diophatine equation:

$$\sum_{i=1}^{3}\frac{1}{p_i} = \frac{1}{5040}. \tag{5.34}$$

If the principal branch corresponds to $(-2, y_{01})$, then the resonances r_i, $i = 1, 2, \ldots, 5$ are positive distinct integers and thus p_1 is a positive integer. Equation (5.30) yields

$$p_1 p_2 p_3 = -(5040)^3 \frac{\kappa^2 \mu^2 \nu^2}{y_{01}^2 y_{02}^2 y_{03}^2}. \tag{5.35}$$

Therefore, either p_2 or p_3 is a negative integer. $\sum r_i = 28$ and r_i being distinct positive integers imply that there are 57 possible values of p_1. Then, one should find all integer solutions (p_2, p_3) of (5.34) for each possible value of p_1. There are 3740 possible integer values of (p_1, p_2, p_3) such that $p_1, p_2 > 0$ and $p_3 < 0$. Equations (5.24), (5.29) and

$$\begin{aligned} &\sum_{i\neq j} r_i r_j = 323 - a_1 y_{01} \qquad \sum_{i\neq j\neq k} r_i r_j r_k = -[(15a_1 + 2a_2) y_{01} - 1988] \\ &\sum_{i\neq j\neq k\neq l} r_i r_j r_k r_l = -a_4 y_{01}^2 - 2(43a_1 + 10a_2 + 6a_3) y_{01} + 7092 \end{aligned} \tag{5.36}$$

determine all the coefficients of (5.2a) in terms of a_1 for all possible values of (p_1, p_2, p_3). Now one should find the roots $\tilde{r}_i, \hat{r}_i$ of (5.2a). There are only three cases such that $\tilde{r}_i, \hat{r}_i$ are distinct integers. The cases and the corresponding simplified equations are as follows.

Case 1:

$$\begin{aligned} &y_{01} = \frac{36}{a_1} \qquad (r_1, r_2, r_3, r_4, r_5) = (2, 3, 4, 9, 10) \\ &y_{02} = \frac{252}{a_1} \qquad (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5) = (-5, -7, 10, 12, 18) \\ &y_{03} = \frac{72}{a_1} \qquad (\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4, \hat{r}_5) = (-2, 3, 5, 10, 12) \\ &a_2 = \tfrac{5}{3} a_1 \qquad a_3 = \tfrac{5}{6} a_1 \qquad a_4 = a_5 = -\tfrac{5}{18} a_1^2 \qquad a_6 = \tfrac{5}{648} a_1^3 \\ &y^{(6)} = a_1 (y y^{(4)} + \tfrac{5}{3} y' y''' + \tfrac{5}{6} y''^2 - \tfrac{5}{18} a_1 y^2 y'' - \tfrac{5}{18} a_1 y y'^2 + \tfrac{5}{648} a_1^2 y^4). \end{aligned} \tag{5.37}$$

Case 2:

$$\begin{aligned} &y_{01} = \frac{28}{a_1} \qquad (r_1, r_2, r_3, r_4, r_5) = (2, 4, 5, 7, 10) \\ &y_{02} = \frac{168}{a_1} \qquad (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5) = (-3, -5, 10, 12, 14) \\ &y_{03} = \frac{84}{a_1} \qquad (\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4, \hat{r}_5) = (-3, 2, 7, 10, 12) \\ &a_2 = 2a_1 \qquad a_3 = \tfrac{3}{2} a_1 \qquad a_4 = a_5 = -\tfrac{5}{14} a_1^2 \qquad a_6 = \tfrac{5}{392} a_1^3 \\ &y^{(6)} = a_1 (y y^{(4)} + 2 y' y''' + \tfrac{3}{2} y''^2 - \tfrac{5}{14} a_1 y^2 y'' - \tfrac{5}{14} a_1 y y'^2 + \tfrac{5}{392} a_1^2 y^4). \end{aligned} \tag{5.38}$$

Case 3:

$$\begin{aligned} &y_{01} = \frac{21}{a_1} \qquad (r_1, r_2, r_3, r_4, r_5) = (3, 4, 5, 7, 9) \\ &y_{02} = \frac{336}{a_1} \qquad (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5) = (-5, -11, 12, 14, 18) \\ &y_{03} = \frac{105}{a_1} \qquad (\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4, \hat{r}_5) = (-5, 3, 7, 11, 12) \\ &a_2 = \tfrac{5}{2} a_1 \qquad a_3 = \tfrac{7}{4} a_1 \qquad a_4 = -\tfrac{2}{7} a_1^2 \qquad a_5 = -\tfrac{5}{14} a_1^2 \qquad a_6 = \tfrac{1}{147} a_1^3 \\ &y^{(6)} = a_1 (y y^{(4)} + \tfrac{5}{2} y' y''' + \tfrac{7}{4} y''^2 - \tfrac{2}{7} a_1 y^2 y'' - \tfrac{5}{14} a_1 y y'^2 + \tfrac{1}{147} a_1^2 y^4). \end{aligned} \tag{5.39}$$

For all three cases, the compatibility conditions are identically satisfied.

To obtain the canonical form of the sixth-order Painlevé-type equation, one should add the non-dominant terms with analytic coefficients of z. That is,

$$\begin{aligned} y^{(6)} = {} & a_1 y y^{(4)} + a_2 y' y''' + a_3 y''^2 + a_4 y^2 y'' + a_5 y y'^2 + a_6 y^4 + A(z) y^{(5)} + B(z) y^{(4)} \\ & + C(z) y y''' + D(z) y''' + E(z) y' y'' + F(z) y y'' + G(z) y'' + H(z) y^2 y' + J(z) y y' \\ & + K(z) y'^2 + L(z) y' + M(z) y^3 + N(z) y^2 + P(z) y + R(z). \end{aligned} \tag{5.40}$$

The coefficients $A(z), \ldots, R(z)$ can be determined by using the compatibility conditions at the resonances. Substituting (5.15) into (5.40) gives the recursion relation for y_j. Then, one can find $A, \ldots, R$ such that the recursion relations for $j = r_1, r_2, r_3, r_4, r_5$ are identically satisfied, and hence $y_{r_1}, y_{r_2}, y_{r_3}, y_{r_4}, y_{r_5}$ are arbitrary.

Case 1. By using the linear transformation (2.6), one can set

$$5a_1^2 A + 6a_1 C + 3a_1 E + 18H = 0 \qquad M = 0 \quad a_1 = 36 \tag{5.41}$$

then, $y_{01} = 1$ and $y_1 = 0$. The compatibility conditions at $j = 2, 3, 4, 9, 10$ imply that all the coefficients are zero except

$$G = -\frac{c_1}{6} \qquad N = c_1 \qquad P = c_2 \qquad R = c_3 \tag{5.42}$$

where c_i are arbitrary constants. Therefore, the canonical form for this case is

$$y^{(6)} = 36\left(y y^{(4)} + \frac{5}{3} y' y''' + \frac{5}{6} y''^2 - 10 y^2 y'' - 10 y y'^2 + 10 y^4\right) - \frac{c_1}{6} y'' + c_1 y^2 + c_2 y + c_3. \tag{5.43}$$

Case 2. One can always choose $y_{01} = 1$, and $y_1 = 0$ by setting

$$45a_1^2 A + 42a_1 C + 21a_1 E + 98H = 0 \qquad M = 0 \quad a_1 = 28. \tag{5.44}$$

Then, the recursion relations imply that if, y_2, y_4, y_5, y_7, and y_{10} are arbitrary then all the coefficients are zero except

$$G = -\frac{c_1}{6} \qquad N = c_1 \qquad P = c_2 \qquad R = c_3 z + c_4 \tag{5.45}$$

where c_i are arbitrary constants. Then the canonical form is

$$\begin{aligned} y^{(6)} = {} & 28\left(y y^{(4)} + 2 y' y''' + \tfrac{3}{2} y''^2 - 10 y^2 y'' - 10 y y'^2 + 10 y^4\right) \\ & - \frac{c_1}{6} y'' + c_1 y^2 + c_2 y + c_3 z + c_4. \end{aligned} \tag{5.46}$$

(5.46) can also be obtained by using (1.16).

Case 3. One can always set $y_{01} = 1$, and $y_1 = y_2 = 0$ by choosing

$$\begin{aligned} & 40a_1^2 A + 28a_1 C + 14a_1 E + 49H = 0 \\ & 40a_1^2 B + 42a_1 F + 28a_1 K + 147M = 0 \qquad a_1 = 21. \end{aligned} \tag{5.47}$$

Then, the recursion relations imply that if, y_3, y_4, y_5, y_7, and y_9 are arbitrary then all the coefficients are zero except

$$B = \frac{c_1}{15} \quad F = -c_1 \quad K = -\frac{3}{4} c_1 \quad P = c_2 \quad R = c_3 \tag{5.48}$$

where c_i are arbitrary constants. Then the canonical form is

$$\begin{aligned} y^{(6)} = {} & 21\left(y y^{(4)} + \tfrac{5}{2} y' y''' + \tfrac{7}{4} y''^2 - 6 y^2 y'' - \tfrac{15}{2} y y'^2 + 3 y^4\right) \\ & - \frac{c_1}{15} y^{(4)} - c_1 y y'' - \frac{3}{4} c_1 y'^2 + c_2 y + c_3. \end{aligned} \tag{5.49}$$

In the procedure used to obtain higher-order Painlevé-type equations, the existence of at least one principal branch has been imposed. However, the compatibility conditions at the positive resonances for the secondary branches are identically satisfied for each case. Instead of having positive distinct integer resonances, one can consider the case of distinct integer resonances. In this case it is possible to obtain equations like Chazy's equation (1.9) which has three negative distinct integer resonances. If all the resonances are negative distinct integers then there are no compatibility conditions and hence, no non-dominant term can be introduced in this procedure. Chazy's equation, which is a simplified equation, can be obtained from the second Painlevé equation by using a similar procedure.

Since the simplified version of P_I is a constant coefficient polynomial-type equation, higher-order constant coefficient polynomial types of simplified equations were considered. However, if one starts from $P_{III}, \ldots, P_{VI}$ one gets the higher-order Painlevé-type equations of the form (1.20).

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